

Lecture

Ch. 7 P. 360

(7.1) (7.2) (7.3)

Eigen values & Eigen vectors :

$$A_{2 \times 2} \text{ \& } A_{3 \times 2}$$

$$\Rightarrow (A - \lambda I)X = 0.$$

$$|A - \lambda I| = 0 \Rightarrow \text{characteristic equation}$$

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

so we get $\lambda = \lambda_1,$

$$\lambda = \lambda_1$$

Example:-

Find the Eigen values and Eigen
vectors of

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}$$

Then find A^6 .

Sol.

$$(A - \lambda I) x = 0.$$

$$\begin{bmatrix} -3-\lambda & 4 \\ -5 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 3-\lambda & 4 \\ -5 & 6-\lambda \end{vmatrix} = 0.$$

We get the characteristic equation.

$$(3-\lambda)(6-\lambda)+20=0.$$

$$-18-3\lambda+\lambda^2+20=0$$

$$\lambda^2-3\lambda+2=0.$$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda=1 \quad \lambda=2 \implies \text{Eigen values.}$$

$$\text{at } \lambda=1.$$

$$\begin{bmatrix} -4 & 4 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{must be same lines.}$$

$$-4x_1 + 4x_2 = 0$$

$$4x_1 = 4x_2 \implies \frac{x_1}{x_2} = \frac{1}{1} \implies P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{at } \lambda=2 \implies \begin{bmatrix} -5 & 4 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-5x_1 + 4x_2 = 0$$

$$5x_1 = 4x_2$$

$$\frac{x_1}{x_2} = \frac{4}{5}$$

$$P_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$$

Eigen vectors

$$\lambda = 1, 2$$

$$D = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

diagonal Matrix.

$$P^{-1} = \frac{1}{1} \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

When diagonalizing a Matrix we must

Make Multiplication

$$D = P^{-1} A P \quad \chi(P)$$

$$PD = \underbrace{P P^{-1}}_I A P$$

$$PD = AP \quad \chi(P^{-1})$$

$$PD P^{-1} = A \underbrace{P P^{-1}}_I$$

$$\boxed{A = P D P^{-1}}$$

$$A^2 = A \times A$$

$$A^2 = P \underbrace{D P^{-1} P}_I D P^{-1}$$

$$P D^2 P^{-1} = A^2 \quad \text{similar} \quad A^3 = P D^3 P^{-1}$$

$$\text{General } A^n = P D^n P^{-1}$$

General: $A^n = P D^n P^{-1}$

in case of 3×3 .

$$A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} P^{-1}$$

in case of 2×2 .

$$A^n = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}$$

in the example.

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$$

$$A^6 = P D^6 P^{-1}$$

$$= \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (1)^6 & 0 \\ 0 & (2)^6 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -64 & 64 \end{bmatrix} \begin{bmatrix} 5-256 & -4+256 \\ 5-320 & -4+320 \end{bmatrix}$$

$$= \begin{bmatrix} -251 & 252 \\ -315 & 316 \end{bmatrix} = \underline{\underline{A \times A \times A \times A \times A \times A}}$$

Example:

Prove that the matrix A satisfied the

Characteristic equation and then find

A^{-1} to the previous example.

$$\text{Char. eq.} = \lambda^2 - 3\lambda + 2 = 0.$$

$$A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$A^2 - 3A + 2I = 0$$

$$A \times A$$

$$\begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} - 3 \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} +$$

$$2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -11 & 12 \\ -15 & 16 \end{bmatrix} - \begin{bmatrix} -9 & 12 \\ -15 & 18 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

$$[A^2 - 3A + 2I = 0] \quad A^{-1}$$

$$A^{-1} A A - 3 A^{-1} A + 2 A^{-1} I = 0$$

$$A - 3I + 2A^{-1} = 0 \Rightarrow A^{-1} = -\frac{1}{2}(A - 3I)$$

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = P^{-1} A P = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = P D^4 P^{-1}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 8 & -8 & 6 \\ -8 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example:

Find the eigen value & eigen vector

then find A^4

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol.

$$(A - \lambda I)x = 0.$$

$$\begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\text{let } |A - \lambda I| = 0.$$

Characteristic equation.

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0.$$

$$A^{-1} = -\frac{1}{2} (A - 3I) \quad \text{or} \quad \left[-\frac{1}{2} (3I - A) \right]$$

$$A^{-1} = \frac{1}{2} \left\{ 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \right\}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -4 \\ 5 & 3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -4 \\ 5 & 3 \end{bmatrix}$$

at $\lambda = 0$.

$$p_2 \equiv p_3 \\ p_1 \equiv p_2$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$-x_1 + x_2 + 0x_3 = 0$$

$$\text{let } x_2 = t, x_3 = s \Rightarrow x_1 = t.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s$$

$$p_2 \checkmark$$

$$p_3 \checkmark$$

$$p_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

